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Local ADHM construction and holomorphic local vector bundles on the twistor space

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Abstract

Local ADHM theory has been discussed; after making some general remarks about Penrose transform and methods of monad, we construct holomorphic vector bundles on the neighbourhood of a projective line in the twistor space. By inverse Ward transformation this corresponds to local solution space of self-dual Yang–Mills equation. In the final section we discuss some possible applications of this theorem.

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1. Introduction

Suppose E is a Hermitian vector bundle over a compact Riemannian four-manifold and E has a unitary connection ∇ whose curvature is the fundamental form F, a two-form with values in the endomorphism bundle of E, i.e.

$$F \in \wedge^2(M) \otimes End(E).$$

The Riemannian metric allows us to decompose F into two components F_+ and F_-, due to the Hodge decomposition of $\wedge^2(M)$. The total energy of the field F is given by the Yang–Mills action

$$YM(F) = \int_M \|F\|^2 \, \mathrm{d}\mu = -\int_M \mathrm{tr}(F \wedge^* F) \, \mathrm{d}\mu.$$

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The Euler–Lagrange equation for this action gives us the Yang–Mills equation

$$\nabla \wedge^{*} F = 0.$$

The conformal invariance of the Hodge star $*$ operator on $\wedge^2$ shows that Yang–Mills equations are conformally invariant in four dimensions. The quantity

$$\int_M \mathrm{tr}(F \wedge F) = \int_M (\|F_-\|^2 - \|F_+\|^2)\,\mathrm{d}\mu$$

is a topological invariant of the bundle E, whose value is $8\pi^2 k$, where k is the characteristic number $c_2 - \frac{1}{2}c_1^2$. This action will be a minimum when either

$$F_+ = 0 \text{ i.e. } {}^*F = -F \quad \text{or} \quad F_- = 0 \text{ i.e. } {}^*F = F,$$

depending on whether $k \geq 0$ or $k \leq 0$, such connections are called anti-instanton or instanton, respectively. From the Bianchi identity, $\nabla \wedge F = 0$, one can readily see that instantons satisfy the Yang–Mills equation.

When G is $SU(2)$ and $k = 1$, the spherically symmetric solutions about the origion in $\mathbb{R}^4$ were discovered by Belavin et al. [BPST75]. For $k > 1$, this has been extended by 'tHooft (unpublished) and Jackiw et al. [JNR77]. These solutions can be imagined as superpositions of k single instantons located at different points of $\mathbb{R}^4$ and the superpositions are achieved through some ansatz. But this ansatz failed to yield solutions for general k instantons. Penrose twistor theory ([At79,WW90]) provides a complete solution of the instanton problem for all classical groups.

The Penrose fibration (see [PR84,PR86,At79,BE89,WW90])

$$\pi : \boldsymbol{CP}^3 \to \boldsymbol{S}^4$$

tells us that each point of S^4 corresponds to $\boldsymbol{CP}^1$ in $\boldsymbol{CP}^3$ and the anti-self-dual (or self-dual) solution of the Yang–Mills equation in the conformally compactified Euclidean 4-space in S^4 corresponds to certain global real algebraic bundles on the complex projective space $\boldsymbol{CP}^3$. The Atiyah–Ward correspondence [At79,Wa77] says, giving an $SU(2)$ anti-instanton (solutions of anti-self-dual Yang–Mills equation) bundle on S^4 is equivalent to giving a homomorphic rank-2 vector bundle ε whose restriction to each projective line is trivial and carries a suitable real structure. In a celebrated paper Atiyah et al. [AHMD78] have shown using Ward correspondence and algebro-geometric techniques 'methods of monads' introduced by Horrocks and Barth [OSS80] that all instantons have a unique description in terms of linear algebra for any arbitrary compact classical group.

Soon after the discovery of (global) ADHM construction [AHMD78] Hartshorne [Ha78] put forward a list problems about the algebraic vector bundles on projective spaces. In that list he also stated the problem of local ADHM as the problem of understanding vector bundles on a tubular neighbourhood of a projective line in the twistor space $\boldsymbol{CP}^3$. As a hint he stated that this problem could also be tackled via Penrose transformations. The local problem is different from the global problem in a number of ways: for example one loses the second Chern class and the moduli space becomes infinite dimensional.

Our main result is:

Theorem 1. *Let E be a vector bundle defined locally on the neighbourhood of a projective line L in $\boldsymbol{CP}^3$ such that the bundle E is trivial when it is restricted to the line. Then bundle E is realized from the cohomology of the following monad*

$$V(-1) \xrightarrow{a} W \xrightarrow{b} U(1),$$

where

$$V = H^1(\Omega^2(1)/H^1(E(-2)), \qquad W = H^1(E \otimes \Omega^1), \qquad U = H^1(E(-1)),$$

i.e. $E = \mathrm{Ker}\, b/\mathrm{Im}\, a$

2. Preliminaries

In this section we discuss some basic features of twister geometry ([PR84,PR86,WW90]) connected to our problem and some definitions regarding monads. So for convenience we split this section into two parts, first part is related to twistor theory and the second one deals with methods of monads.

2.1. Some features of twistor theory

The idea of twistor theory is quite old and goes back to the famous Plücker–Klein relationship [WW90] where it describes the straight lines in $\boldsymbol{CP}^3$ by the points of a quadric hypersurface Q in $\boldsymbol{CP}^5$. In the Penrose twistor programme one uses the holomorphic geometry of the twistor space to produce solutions to differential equations. Recall the Penrose fibration defined by $\pi : \boldsymbol{CP}^3 \longrightarrow \boldsymbol{S}^4$ with fibre $\pi^{-1}(x)$ at each point $x \in \boldsymbol{S}^4$ is $\boldsymbol{CP}^1$ which precisely gives the compatible complex structure in $\boldsymbol{TS}^4$. We can pull back an $SU(2)$ bundle $\tilde{E}$ on $\boldsymbol{S}^4$ by π to obtain an associated rank-2 bundle E on $\boldsymbol{CP}^3$. The connection ∇ on $\tilde{E}$ is anti-self-dual if and only if the pulled back connection determines a holomorphic structure on $E = \pi^*(\tilde{E})$. This is the basis of the Ward transformation [Wa77,Wa90]. A connection with anti-self-dual curvature on the original $SU(2)$ bundle gives an almost complex structure on E and the anti-self-duality condition provides the integrability condition needed for E to be a holomorphic rank-2 vector bundle on $\boldsymbol{CP}^3$ and since the bundle comes from the bundle over $\boldsymbol{S}^4$ it carries a real structure. The relevant anti-holomorphic involution is given by $k : \boldsymbol{CP}^3 \longrightarrow \boldsymbol{CP}^3$,

$$k(z_1, z_2, z_3, z_4) = (-\bar{z}_2, +\bar{z}_1, -\bar{z}_4, +\bar{z}_3),$$

where $z_i s$ are homogeneous coordinates on $\boldsymbol{CP}^3$. This map is conjugate linear in the sense that $k(\lambda z) = \bar{\lambda} k(z)$ for any $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^4$. Each fibre $\pi^{-1}(x)$ is a k-invariant projective line and the restriction of the pull-back bundle E on each real line $\pi^{-1}(x)$ is trivial. Also one can easily see that the induced automorphism of the space of lines can be realized as the complex conjugate of Plücker coordinates of the quadric, thus real points $\mathcal{Q}_R$ of the quadric $\mathcal{Q}$ correspond to real lines in $\boldsymbol{CP}^3$ (cf. [WW90]).

So that holomorphic vector bundles E coming from instantons over $\boldsymbol{CP}^3$ have zero first Chern class (which is clear since the structure group is $SU(2)$ so $\det E$ is trivial) and the instanton number k is the second Chern class $c_2(E)$. We know from GAGA [Se56] that all holomorphic bundles on the projective spaces have unique algebraic structures. Fixing $c_1 = 0$ and $c_2 = k$ (say) we can define the moduli space M_k of stable algebraic rank-2 vector bundles on $\boldsymbol{CP}^3$. The bundles coming from instantons have some characteristic features which we will discuss in Section 2.2.

2.2. *Methods of monads*

There are two main ways of studying vector bundles over complex projective spaces. One is via curves and jumping line, the other is by monads [OSS80]. The idea is twist the bundle E by $\mathcal{O}(n)$ so that the new bundle $E(n)$ has plenty of global sections. If s is a generic section then the set of points in $\boldsymbol{CP}^3$ where s becomes zero will be a curve, C, in $\boldsymbol{CP}^3$. With the given curve and some algebraic data and machinery one can recover E. The second method is the most seccessful and widely used technique.

A monad is a pair of maps of holomorphic vector bundle over a complex manifold $\mathcal{M}$,

$$L(-1) \xrightarrow{a} M \xrightarrow{b} N(1),$$

such that a is injective and b is surjective and the composite map $ba = 0$ everywhere. The bundle $E = \operatorname{Ker} b / \operatorname{Im} a$ is the 'cohomology' of the monad. If the rank of the vector bundle is k then the dimensions of the L, M and N vector spaces would be n, $2n + k$ and n, respectively.

The word monad was used by Horrocks. The idea of this method is to constuct complicated bundles from three simpler bundles L, M and N over the $\mathcal{M}$. The process of taking cohomology of a complex is in general functorial, so that two monads which are isomorphic (in the categorical sense) define isomorphic vector bundles.

In order to see how the connection arises from the monad we shall follow Donaldson [Do85]. Let X be the two-dimensional vector space underlying $\boldsymbol{P}(X) = \boldsymbol{P}^1$. Here '$a$' is an element of $X^* \otimes \operatorname{Hom}(L, M)$ and 'b' is an element of $X^* \otimes \operatorname{Hom}(M, N)$.

Composition defines an element 'c' of $X^* \otimes X^* \operatorname{Hom}(L, N)$. Since $ba = 0$ is satisfied everywhere, c is skew-symmetric on X^*. The condition that the bundle E be holomorphically trivial on the projective line is that

$$\wedge^2 X^* \otimes \operatorname{Hom}(L, N) \cong \operatorname{Hom}(L, N)$$

is an isomorphism.

Following Donaldson, this triviality condition can be re-expressed by choosing two distinct points m, n in the projective line. Thus we obtain four linear subspaces of the vector space M, given by

$\operatorname{Im} a_m$	$\ker b_m$
$\operatorname{Im} a_n$	$\ker b_n$

Elementary linear algebra shows that the restriction of E is naturally isomorphic to the subspace

$$\ker b_m \cap \ker b_n \subset M$$

and also to the quotient,

$$M/\operatorname{Im} a_m + \operatorname{Im} a_n.$$

The above two descriptions mean that the fibre of the associated bundle E' comes as a projective subspace of the fixed vector space M equipped with maps

$$E' \overset{i}{\rightleftharpoons} M.$$

Now using these projective maps we know to define connection of a subbundle of a fixed vector space. Suppose M has the flat connection ∇ and we have a smooth bundle projection $\pi : M \longrightarrow E'$, which is a left inverse to the inclusion map i. Then we get an induced connection A on E' with covariant derivative

$$\pi \circ \nabla \circ i(s).$$

Thus we get a connection on any bundle E associated to monad on the twistor space.

In principle the use of monads reduces the study of vector bundles to linear algebra. Once we obtain a vector bundle from the monad then the inverse Ward correspondence [Wa77,Wa90] gives the general ADHM description of all self-dual gauge fields over S^4.

When the bundles have some additional structure then this additional structure goes into the monad automatically [Ha79]. Let $\mathcal{E}$ be the sheaf of holomorphic (or algebraic) sections of $\pi^*(E)$ over $\boldsymbol{CP}^3$. Suppose the coherent sheaves have the following vanishing cohomologies:

$$H^0(\mathcal{E}(m)) = 0 \quad \text{for } m \le -1, \qquad H^1(\mathcal{E}(m)) = 0 \quad \text{for } m \le -2,$$
$$H^2(\mathcal{E}(m)) = 0 \quad \text{for } m \ge -2, \qquad H^3(\mathcal{E}(m)) = 0 \quad \text{for } m \ge -3.$$

The coherent sheaves on $\boldsymbol{CP}^3$ with these properties are called admissible sheaves (see [MD78,Ha78]) and the corresponding monad will be a special monad. There is a functorial equivalence between the category of special monads and the category of admissible sheaves. In order to prove the vanishing of these cohomologies it suffices to show the vanishing of the first two cohomologies. The other two follow form the Serre duality,

$$H^i(\varepsilon(m))^* \cong H^{3-i}(\mathcal{E}^*(-4-m)).$$

The fibres of $\pi : \boldsymbol{CP}^3 \longrightarrow \boldsymbol{S}^4$ are the projective lines in $\boldsymbol{CP}^3$ and the restriction of ε to them is holomorphically trivial and for that reason $H^0(CP^1, \mathcal{O}(m)) = 0$ when $m < 0$, hence $H^0(CP^1, \mathcal{E}(m)) = 0$.

Since in the entire calculation we have used the local version of Beilinson's spectral sequence ([Be78,OSS80]). To keep the paper self-concise we give the statement of the theorem without proof (for proof see [OSS80]).

Theorem 2 (see [Be78,OSS80, 3.1.4]). *Let E be an m-dimensional holomorphic bundle over the Zariski open subset U of $\boldsymbol{CP}^n$ then there exists a spectral sequence E_m^{pq} with E_1-term*

$$E_1^{pq} = H^q(U, E \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}_{P_n}(q),$$

which converges to

$$E^j = E \quad for\ j = 0$$

and otherwise 0. This means that

$$E_\infty^{pq} = 0 \quad for\ p + q \neq 0$$

and

$$\bigoplus_{p=0}^{n} E_\infty^{-p,p}$$

is the associated graded sheaf of a filtration of E.

Beilinson's work has enabled us to construct an inverse functor, i.e. it helps us to construct monads from the admissible sheaves. Consider for example the global ADHM case [AHMD78], the monads coming from instantons always have a special structure

$$A(-1) \xrightarrow{\alpha} B \xrightarrow{\beta} C(1),$$

where $A(-1) = A \otimes \mathcal{O}(-1)$, etc. and A, B and C are three complex vector spaces. Barth observed that corresponding bundles $\mathcal{E}$ on $\boldsymbol{CP}^3$ with $c_1 = 0$ and $c_2 = k$ are stable and satisfy $H^1(\mathcal{E}(-2)) = 0$, using Penrose transform we can deduce that it is equivalent to the condition that there are no non-zero solutions of the equation.

$$(\Delta + \tfrac{1}{6}R)s = 0$$

has no global solutions. Here Δ denotes Laplace–Beltrami operator coupled to the connection, $R > 0$ is the positive scalar curvature of $\boldsymbol{S}^4$ and s is a section of $\tilde{E}$.

But in the local case this vanishing argument does not apply so the cohomology group $H^1(\mathcal{E}(-2)) \neq 0$ in the local case; moreover, this will appear in the vector spaces of the monad and since the bundle is supported on a non-compact space we cannot use Serre duality either. Instead of that we will use some techniques of several complex variables to deduce the vanishing of the cohomologies in the spectral sequence.

3. Construction of monads for local bundles

In this section we construct the monads of holomorphic bundles on a tubular neighbourhood of a projective line in $\mathbf{CP}^3$. It has been known that Penrose transformation deals with double fibration of a generalized flag variety [BE89]. This transformation has been used in the local ADHM problem by localization at a point in $\mathbf{S}^4$ which corresponds to localization near a line in $\mathbf{CP}^3$. Let us recall the basic double fibration

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{f} & \mathcal{Q} \\ \downarrow{\scriptstyle g} & & \\ \mathbf{CP}^3 & & \end{array}$$

where $\mathbf{F}$ is the flag variety and $\mathcal{Q}$ the complexification of $\mathbf{S}^4$. As we choose the image variety of the transform a *Stein* subset [GR77] of the complexification of $\mathbf{S}^4$.

Definition 3. A closed subset in V of a complex space X is called Stein set (in X) if Cartan's 'theorem B' holds good. This says for every coherent analytic sheaf Ξ

$$H^q(V, \Xi) = 0 \quad \text{for all } q \geq 1$$

is valid on V. A complex space which is itself a Stein set is called a stein space.

Given a Stein subset of $\mathcal{Q}$ and with the help of the map f we can pull back this Stein set to flag variety. Let S^a be the Stein subset of $\mathcal{Q}$. Suppose $\mathbf{F}^a = f^{-1}\mathbf{S}^a$ is the open subset of flag variety then by pushing down this open subset we obtain the corresponding open subset of the twistor space $\mathbf{P}^a$. So the basic double fibration induces a double fibration among the open subset

$$\begin{array}{ccc} \mathbf{F}^a & \xrightarrow{f} & \mathbf{S}^a \\ \downarrow{\scriptstyle g} & & \\ \mathbf{P}^a & & \end{array}$$

Here $\mathbf{P}^a$ is the open subset of the twistor space and this can be covered by two Stein subsets, i.e.

$$\mathbf{P}^a = (\mathbf{P}^a\{southpole\}) \cup (\mathbf{P}^a\{northpole\}).$$

In order to see this, let us take a small thickening of a projective line, basically this yields the neighbouhood of a line. We have to show thickenings can be covered by thickenings of two Stein subsets.

Let us consider another projective line (say l') disjoint from the first one (say l). Let us consider a family of planes through l'. Let n_l be the neighbourhood of a line l. This family defines a map,

$$n_l \longrightarrow l.$$

Suppose the fibre of this map is Stein, if we restrict the base to (CP_1-point), we get a fibration by the family of planes.

Total space is the fibration over a Stein manifold where fibres are Stein. Then by Leray Spectral sequence one can see that higher cohomologies vanish.

Alternatively, one can check this in terms of some explicit coordinates too (w, z) and $(\tilde{w}, \tilde{z})$, where on the overlapping neighbourhoods, $\tilde{z} = z^{-1}$ and $\tilde{w} = wz^{-1}$,

One can get a smaller neighbourhood

$$|z| < r|w| < R, \qquad |\tilde{z}| < r|\tilde{w}| < R_r,$$

so that each open set is a product of discs and the intersection of the product of an annulus and disc.

In the case of Stein subset which is cut out from the centre, the first cohomology does not vanish.

Let Y_1 and Y_2 be the two open centrally cut out Stein subsets of $\boldsymbol{P}^a$ and let $\boldsymbol{P}^a = Y_1 \cup Y_2$ such that $H^1(Y_i, \Xi) \neq 0$.

Lemma 4. *Let X be a complex space and V_1 and V_2 be two Stein subspaces of X. Then $V_1 \cap V_2$ is Stein too.*

For proof one must consult Grauert and Remmert [GR77].

Proposition 5. *Let $\boldsymbol{P}^a$ be the open subset of twistor space constructed above. Every coherent analytic sheaf $\mathcal{F}$ on $\boldsymbol{P}^a$ satisfies*

$$H^q(\boldsymbol{P}^a, \mathcal{F}) = 0$$

for all $q \geq 2$.

Proof. Since $\boldsymbol{P}^a = Y_1 \bigcup Y_2$ using Mayer–Vietoris sequence [BT82] we obtain

$$\longrightarrow H^{q-1}\left(Y_1 \bigcap Y_2, \Xi\right) \longrightarrow H^q(\boldsymbol{P}^a, \Xi) \longrightarrow H^q(Y_1, \Xi) \oplus H^q(Y_2, \Xi) \longrightarrow \cdots$$

By theorem B of H. Cartan we already know

$$H^q(Y_i, \Xi) = 0 \quad \text{for } q \geq 1$$

and

$$H^q(Y_1 \cap Y_2, \Xi) = 0$$

by lemma 4. So $H^q(\boldsymbol{P}^a, \Xi) = 0$ for any coherent sheaf Ξ when $q > 1$. This completes the proof. □

The result of this proposition will be used to establish the vanishing of relevant cohomology groups in the spectral sequence.

Now we follow the procedure of Drinfeld and Manin in [MD78] where they have given a nice procedure of constructing vector bundle using monad. Let Ω^1 denote the cotangent bundle of CP^3 and Ω^n the corresponding nth exterior product of the cotangent bundle. We obtain the following sheaf exact sequence

$$[T_{CP^3}(-1)]^\vee \longrightarrow \mathcal{O}_{CP^3} \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Resolving into locally free modules we obtain the following Koszul complex in our case

$$\Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \mathcal{O}_{CP^3} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Following Drinfeld–Manin [MD78] we tensor the above sequence with an arbitrary vector bundle $E(-1)$ so that we obtain the following exact sequence from the Koszul complex

$$\Omega^3(2) \otimes E \longrightarrow \Omega^2(1) \otimes E \longrightarrow \Omega^1 \otimes E \longrightarrow E(-1)|_{CP^3} \longrightarrow E(-1)|_P \longrightarrow 0.$$

In order to extract the information of the bundle E we have to go for the spectral sequence developed by Beilinson [Be78]. In our case the spectral sequence associated with the double complex would be the following one:

$H^3(\Omega^3(2) \otimes E)$	$H^3(\Omega^2(1) \otimes E)$	$H^3(\Omega^1 \otimes E)$	$H^3(E(-1))$
$H^2(\Omega^3(2) \otimes E)$	$H^2(\Omega^2(1) \otimes E)$	$H^2(\Omega^1 \otimes E)$	$H^2(E(-1))$
$H^1(\Omega^3(2) \otimes E)$	$H^1(\Omega^2(1) \otimes E)$	$H^1(\Omega^1 \otimes E)$	$H^1(E(-1))$
$H^0(\Omega^3(2) \otimes E)$	$H^0(\Omega^2(1) \otimes E)$	$H^0(\Omega^1 \otimes E)$	$H^0(E(-1))$

The two cohomologies are related by the operators satisfying

$$d_r\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

such that $d_r^2 = 0$. When $r = 1$ we have the cohomology of the rows above.

Our strategy is to find the monad corresponding to this bundle E. This is possible provided sufficient number of cohomology groups are zero in the spectral sequence. The vanishing of higher cohomologies follows from our earlier result (Proposition 5)

$$H^3(\Omega^3(2) \otimes E) = H^3(\Omega^2(1) \otimes E) = H^3(\Omega^1 \otimes E) = H^3(E(-1)) = 0,$$

and

$$H^2(\Omega^3(2) \otimes E) = H^2(\Omega^2(1) \otimes E) = H^2(\Omega^1 \otimes E) = H^2(E(-1)) = 0.$$

Therefore, we conclude that the first two rows of the spectral sequence vanish identically. Our next task is to show that the bottom most row also vanishes.

Lemma 6. *Let E be the bundle defined on the tubular neighbourhood of projective line. If it is trivial on the line then it satisfies* $H^0(E(-k)) = 0$ *for all* $k > 0$.

Proof. This is a trivial case of Kodaira vanishing theorem [GH78], hence we obtain

$$H^0(\mathcal{O}(-k)) = 0$$

for all $k > 0$. So the result follows immediately. □

Claim 7.

$$H^0(\Omega^3(2) \otimes E) = H^0(\Omega^3(1) \otimes E) = H^0(\Omega^1 \otimes E) = H^0(E(-1)) = 0.$$

Proof. Restricting to a line the tangent bundle of the $\boldsymbol{CP}^3$ fits into the exact sequence

$$\mathcal{O}(2) \longrightarrow T_{CP^3|L} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Hence we obtain the following splitting of the tangent bundle

$$T_{CP^3|L} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Then the dual of this splitting will be the splitting of the cotangent bundle.

$$[T_{CP^3}]^\vee = \Omega^1 = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$
$$E \otimes \Omega^1|_L = E(-2) \oplus E(-1) \oplus E(-1).$$

So we conclude from the previous lemma

$$H^0(\Omega^1 \otimes E) = H^0(E(-1)) = 0.$$

Since $\Omega^3 = \mathcal{O}(-4)$, i.e. the canonical bundle of $\boldsymbol{CP}^3$, the first cohomology group is reduced to

$$H^0(\Omega^3(2) \otimes E) \cong H^0(E(-2)) = 0.$$

Hence we obtain $H^0(\Omega^2(1) \otimes E) = 0$ from the spectral sequence. Then in the spectral sequence the whole 0th row vanishes. Thus we prove the lemma. □

Thus we are left with second row only which is expressed as follows:

$$H^1(\Omega^3(2) \otimes E) \longrightarrow H^1(\Omega^2(1) \otimes E) \longrightarrow H^1(\Omega^1 \otimes E) \longrightarrow H^1(E(-1)).$$

Observe that the first element in the sequence is $H^1(\Omega^3(2) \otimes E = H^1(E(-2))$, since Ω^3 stands for canonical line bundle of $\boldsymbol{CP}^3$ and hence $\Omega^3 = \mathcal{O}(-4)$. This element vanishes identically in the case of bundle over $\boldsymbol{S}^4$, but in the local case it contributes to the monad. This sequence of four vector spaces can be easily transformed into standard monad, i.e. a pair of morphisms and three vector spaces. The monad of the local bundle E is

$$[H^1(\Omega^2(1) \otimes E)/H^1(E(-2))] \otimes \mathcal{O}(-1) \xrightarrow{a} H^1(\Omega^1 \otimes E)$$
$$\xrightarrow{b} [H^1(E(-1))] \otimes \mathcal{O}(1),$$

where a and b are two morphisms and the bundle is recovered from the cohomology of the monad.

Remark 8. If we compare our chapter with the earlier paper of Witten [Wi79] (he attempted this calculation for Minkowski space time) we find the following replacements: (1) the first vector space is the quotient space $H^1(\Omega^2(1) \otimes E)/H^1(E(-2))$, not the space

$H^1(\Omega^2(1) \otimes E)$ which was found by Witten. (2) Witten used a long exact sequence, which is wrong, instead of Beilinson's spectral sequence. Moreover, he did not show explicitly why the cohomologies vanished. (3) Moreover, we want to point out that unlike in the global case, the vector space A is not dual to C in the local case.

Putting all the results concerning monad and local vector bundle together we obtain our main theorem.

4. Identification of the cohomologies

In this section we will identify the vector spaces appearing in the monads. In the local case all the vector spaces forming the monads are infinite dimensional vector spaces. They are the solutions of the three auxiliary equations as Witten showed. He showed in the first half of his paper that two of the vector spaces are the solutions of Dirac equations and the other one is the solution of some scalar equation.

In order to see this in detail we must apply the Penrose transform. In this section we will demonstrate how to obtain the information about $H^1(\Omega^1)$.

This approach is based on local twistor theory as shown by Lionel Mason [Ma87]. We will use spinorial approach [WW90].

Now we denote, $S = \mathcal{O}_{A'}$ and $S' = \mathcal{O}^A$ and their dual are $S^* = \mathcal{O}^{A'}$ and $S'^* = \mathcal{O}_A$.

So the tangent bundle

$$T = S' \otimes S = \mathcal{O}^{AA'}.$$

Hence its dual is

$$\Omega^1 = S'^* \otimes S = \mathcal{O}_{AA'}.$$

Naturally,

$$\Omega^2 = [sym^2 S'^*] \otimes [\wedge^2 S \oplus \wedge^2 S'^*] \otimes [sym^2 S].$$

The bundle $L = \wedge^2 S'$ is called determinant line bundle on Q. If we fix the element of $\wedge^4 C^4$ then we can indentify $L = \wedge^2 S'^*$. In the notation $L = \mathcal{O}[1]$ and $L^* = \wedge^2 S'^* = \mathcal{O}[-1]$. Hence we can write

$$\Omega^2 = \mathcal{O}_{(AB)}[-1] \oplus \mathcal{O}_{(A'B')}.$$

Let us consider the Euler sequence on the twistor space

$$0 \longrightarrow \mathcal{O} \xrightarrow{\times Z^\alpha} \mathcal{O}^\alpha(1) \longrightarrow T \longrightarrow 0.$$

One can regard Z^α as the tautological section. Dualizing the above sequence, we obtain

$$0 \longrightarrow \Omega^1 \longrightarrow \mathcal{O}_\alpha(-1) \xrightarrow{\times Z^\alpha} \mathcal{O} \longrightarrow 0.$$

So from the long exact sequence we obtain

$$0 \longrightarrow H^0(\boldsymbol{P}^a, \mathcal{O}) \longrightarrow H^1(\boldsymbol{P}^a, \Omega^1) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}).$$

Penrose transformation of $H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1))$ satisfies

$$\nabla^B_{B'}\phi_{B\alpha} = 0,$$

where ∇ is the spin connection and α the twistor index.

The definition of the local twistor and their construction then give us that $\phi_{B\alpha}$ is equivalent to a pair of fields $\xi_{BA'}$, η_{BA} and these are the sections of $\mathcal{O}_{BA'}$ and $\mathcal{O}_{BA}[-1]$, respectively. These satisfy

$$\nabla^B_{B'}\xi'_{BA} - i\epsilon_{B'A'}\eta^B_B = 0, \qquad \nabla^B_{B'}\eta_{BA} = 0.$$

This tells us ξ'_{BA} is the potential for the left-handed Maxwell field.

$$0 \longrightarrow H^0(\boldsymbol{P}^a, \mathcal{O}) \longrightarrow H^1(\boldsymbol{P}^a, \Omega^1) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}).$$

Then $H^1(\boldsymbol{P}^a, \mathcal{O})$ is isomorphic to potentials modulo gauge for such fields.

We are interested in

$$0 \longrightarrow H^0(\boldsymbol{P}^a, \mathcal{O}) \longrightarrow H^1(\boldsymbol{P}^a, \Omega^1) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}),$$
$$0 \longrightarrow H^1(\boldsymbol{P}^a, \Omega^1)/H^0(\boldsymbol{P}^a, \mathcal{O}) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O}).$$

We want to seek the kernel of the map $H^1(\boldsymbol{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\boldsymbol{P}^a, \mathcal{O})$.

$$\xi'_{BA} = \nabla_{BA'}f$$

for some function f. If we insert this into the first equation we obtain

$$\nabla^B_{B'}\nabla_{BA'}f - i\epsilon_{B'A'}\eta_{BA} = 0.$$

This turns out to be

$$\Box f = \eta^B_B.$$

Then applying $\Box$ once again we obtain

$$\Box^2 f = 0.$$

Similarly, the cohomology groups have been identified by Lionel Mason and Mike Singer in an unpublished article [MS87], these exactly coincide with the Witten complex as predicted by Witten [Wi79].

$$A \xrightarrow{a(z)} B \xrightarrow{b(z)} C,$$

where z denotes homogeneous coordinates of CP^3. We decompose z into two spinors,

$$z^i = (\eta A, \chi^{A'}), \text{ say,}$$

as usual $A, A' = 1, 2$. In this notation,the anti-self-dual plane corresponding to z is defined by

$$\pi^{A'}(x) := \chi^{A'} - x^{AA'}\eta_A.$$

Description of spaces:

Space C. This is spanned by solutions of Dirac equation

$$D_{AA'}\psi^{A'} := 0.$$

Let us define a spinorial derivative [PR84]

$$d_w^{A'} := D^{AA'}w_{A'}$$

with this notation we can always write the above equation

$$d_w^{A'}\psi^{B'} - d_w^{B'}\psi^{A'} = 0,$$

so that the row vector $\psi^{A'}$ will form the basis of C.

Space B. The elements of B are given in terms of complete basis $(\phi, \omega_\alpha^\beta)$ forming row vectors of linearly independent solutions of

$$d_w^{A'}\phi = \omega_\alpha^{A'} z^\alpha, \tag{$*$}$$

where $z^\alpha = (w, xw)$.

Since space C is complete,

$$\omega_\alpha^{A'} = \psi^{A'} B'_\alpha.$$

Then RHS of $(*)$ is

$$\psi^{A'}[B'_\alpha z^\alpha] = P'^A(x)u_A,$$

where

$$P'^A = c_1'^A + c'_{2A}x^{AA'} \quad \text{and} \quad B'_\alpha = (c_1'^A, c'_{2A}).$$

Hence we obtain

$$D^{AA'}\phi = \psi^{A'}P'^A.$$

Space A. The elements of space A are the solutions of

$$D_{AA'}D^2\eta^{A'} = 0.$$

Please note that

$$D_{AA'}D^{A'B} = D^2\delta_A^B.$$

Remark 9. The space $H^1(E(-2))$ is exactly 4th vector space and $\ker a$ (as shown by Witten [Wi79, p.217]).

$\operatorname{Im} a$. Let $(\phi v, \omega_\alpha v) \in [\operatorname{im} f(z)]$ and if we take $v = B(z)w$ for some w. Then $\phi v = \lambda_{B'} w \pi^{B'}$, where we used $\lambda_\alpha = \phi B_\alpha$ and $\eta^B = \eta_{B'} x^{BB'}$, and the definition of anti-self-dual plane.

Hence, Im a consists of scalar fields that vanish on anti-self-dual plane.

Similarly, Witten showed that, ker g consists of scalar fields that are convariantly constant on the ASD plane.

Remark 10. The first part of Witten's paper completely agrees with our result.

Now we are in the position to lay out explicitly the local ADHM theorem. First we must define the data of local ADHM construction which we have already gathered from the last two sections.

Data (Local):

(1) Three infinite dimensional vector spaces A, B and C where A and C are the solution spaces of Dirac equations and B is the solution space of scalar equations.
(2) D is another vector space formed by the solution of the Laplace equation on S^a.
(3) The quotient space A/D, solutions of Dirac equation modulo the harmonic solution.
(4) Two linear maps a and b, where a: $A/D \longrightarrow B$ is an injective map and b: $B \longrightarrow C$ is the surjective map and these give us a structure monad. These maps are linear over the complex projective space.
(5) The cohomology of the monad or the quotient space $\operatorname{Im} a/\operatorname{Ker} b$ gives the bundle from the local monad.

Please note that the equivalence classes of monad means equivalence classes of ADHM (local) data and this gives rise to equivalent classes of local vector bundles on the neighbourhood of a line

Theorem 11. *There exists a one to one correspondence between (a) equivalence classes of local ADHM data or the equivalent classes of local vector bundles on the formal completion of the projective line on* $\boldsymbol{CP}^3$, *(b) gauge equivalence classes of local solutions of self-dual Yang–Mills equation.*

5. Applications, discussions and open problem

In this section we have attempted to show some applications of local ADHM, particularly from the point of view of reduction of self-dual Yang–Mills equation. At the end of this section we have focused on some of the interesting problems concerning local vector bundles.

During the last few years Ward, Mason (see for example [Wa90,MS87]) and others have shown that many integrable systems particularly in $1+1$ dimensions are symmetry reductions of self-dual Yang–Mills equation. The motivation of these 'phenomenological' works show that it could be possible to view the self-dual Yang–Mills equation is the universal integrable system. But it is too early to say since so far mathematical physicists have failed to show famous equations like KP or Davey–Stewartson are the reductions of the self-dual Yang–Mills equation. But it would be rather interesting to know how geometry of self-dual Yang–Mills equation is related to the geometry of the reduction equations. Here we picked up KdV as an example to show how its geometry fits with local ADHM construction. We choose to work on $\mathcal{R}^4$ with coordinates (x, y, z, t) and the metric

$$ds^2 = dx^2 - dy^2 - 4\,dz\,dt.$$

The Yang–Mills connection $D := \partial - A$ where A takes values in the Lie algebra of $SL(2, C)$ and these are defined upto gauge transformation

$$A \longrightarrow hAh^{-1} - (\partial h)h^{-1}.$$

Following Belavin and Zakharov [BZ78], the self-duality conditions become

$$[D_x - D_y, D_t] = 0, \qquad [D_x + D_y, D_t] = 0,$$
$$[D_x - D_y, D_x + D_y] + [D_z, D_t] = 0.$$

Then performing two-dimensional reduction, one null and the other time-like and by imposing gauge fixing condition, Mason–Sparling [MS87] and Bakas–Depireux [BD91] showed that SDYM equation reduces to KdV equation.

We have already encountered one-dimensional reduction (here only one non-null translation symmetry along ∂_y is imposed) in the case of Bogomolny equation in $\mathbb{R}^3$ where Hitchin [Hi82] and Nahm [Na81] have shown this is equivalent in $\mathbb{R}^4$, which is in addition invariant under the action of the additive group $\mathbb{R}$ of translation in the z-direction. By means of twistor correspondances Hitchis showed that the $SU(2)$ Bogomolny equation on $\mathbb{R}^3$ corresponds to a holomorphic rank-2 vector bundle E on $\boldsymbol{TP}_1$ which is quaternionic and trivial on every real section of $\pi : \boldsymbol{TP}_1 \longrightarrow \boldsymbol{P}_1$.

In the KdV case we have gone one step further, KdV in $\mathbb{R}^2$ is equivalent to a solution of the self-duality equation in $\mathbb{R}^4$ which is in addition invariant under the action of the additive group $\mathbb{R} + \mathbb{R}$ of translation which is a pair of orthogonal space–time translation one time-like and one null direction. On top of that, satisfies some gauge fixing conditions which we have listed below.

Proposition 12. *If we reduce the self-dual Yang–Mills equation by the pair of two orthogonal Killing vectors (one is space-like and other time-like) ∂_y and ∂_z and fixing the gauge*

$$A_z = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad A_x + A_y = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \quad \textit{and} \quad A_x - A_y = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix},$$

we obtain the KdV equation as the reduction of self-dual Yang–Mills equation.

Let us call this data a reduction data. Recall that the monad of the local vector bundle

$$H^1(\Omega^2(1) \otimes E)/H^1(E(-2)) \xrightarrow{a} H^1(\Omega^1 \otimes E) \xrightarrow{b} H^1(E(-1))$$

and the morphisms a and b are linear over the projective space. Now in the reduced case these morphisms must be two-translation-invariant and the corresponding vector bundle is also two-translation-invariant. As Mason and Sparling [MS87] showed, a solution of $SU(2)$ KdV equation on $\mathbb{R}^2$ corresponds to a holomorphic rank-2 holomorphic vector bundles over $\boldsymbol{TP}^1$ on which we have the action of an additional symmetry which corresponds to extra symmetry.

In the reduction case, one important point should be noted which tells us not every two-translation invariant solution of self-dual Yang–Mills equation are the solutions of KdV, since we have imposed a null translation along ∂_z and the gauge fixing in the same direction. This is finiteness condition which is similar to what Hitchin [Hi90] showed in the harmonic case.

There are some open problems in the case of local vector bundle. As Hartshorne [Ha78] pointed out, a global bundle on $\boldsymbol{CP}^3$ is determined by its restriction to the formal neighbourhood of a projective line so that the local problem gives us a new perspective on the global problem.

There is another celebrated problem in the gauge theory that is also a local problem. This is the construction of vector bundles from the full fledged Yang–Mills sheaves (see [HM80,IYG78, Wi78]) in the neighbourhood of some $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ inside the hypersurface lying inside $\boldsymbol{CP}^3 \times \boldsymbol{CP}^3$.

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